ATKINSON SCIENCE

THEORY GUIDE

Equations of Fluid Flow

Momentum Equations in Cartesian Coordinates

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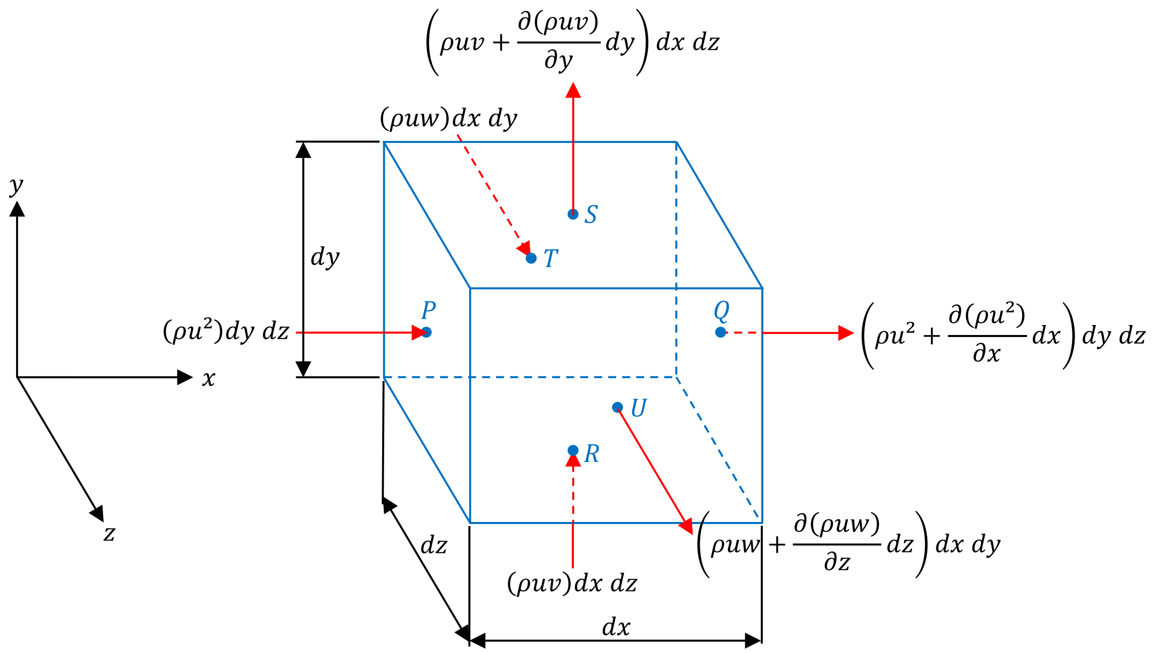
1 Control volume analysis

Momentum is a vector quantity, so there are three momentum components, one for each coordinate direction, and three momentum equations. To derive the equation of, say, the x component of momentum, we apply Newton's second law of motion (the principle of conservation of momentum) to the control volume (CV) shown in Figure 1. When applied to the CV, Newton's second law can be written:

$$\boxed{\begin{array}{l}\text{Rate of increase}\\ \text{of } x \text{ component}\\ \text{momentum of}\\ \text{fluid in CV}\end{array}} = \boxed{\begin{array}{l}\text{Rate of flow of } x\\ \text{component}\\ \text{momentum into}\\ \text{CV}\end{array}} - \boxed{\begin{array}{l}\text{Rate of flow of } x\\ \text{component}\\ \text{momentum out}\\ \text{of CV}\end{array}} + \boxed{\begin{array}{l}\text{Sum of } x\\ \text{components of}\\ \text{forces applied}\\ \text{to fluid in CV}\end{array}} \qquad (1)$$

Figure 1 Infinitesimal control volume for Cartesian coordinates



2 Transient and convection terms

The x component of momentum in the CV is equal to the x component of velocity, u [m s^{-1}], times the mass of fluid in the CV, $\rho\ dx\ dy\ dz$ [kg]; that is, $\rho u\ dx\ dy\ dz$ [kg m s^{-1}]. The rate of increase of x component momentum with time (the first term in Eq. (1)), is therefore

$$\frac{\partial(\rho u)}{\partial t} dx\ dy\ dz \quad [\text{kg m s}^{-2}] \quad (2)$$

x component momentum may enter or leave through any of the faces P to U in Figure 1, transported by the mass flow through the faces.

The rate of flow of x component momentum through the face perpendicular to the x direction whose centre is P is u [m s^{-1}] times the mass flow through the face, $\rho u\ dy\ dz$ [kg s^{-1}]; that is,

$$\rho u^2\ dy\ dz \quad [\text{kg m s}^{-2}]$$

The rate of flow of x component momentum through the opposite face whose centre is Q is

$$\left(\rho u^2 + \frac{\partial(\rho u^2)}{\partial x} dx\right) dy\ dz \quad [\text{kg m s}^{-2}]$$

and so the net rate of flow of x component momentum out of the CV through the faces with centres P and Q is

$$\left(\rho u^2 + \frac{\partial(\rho u^2)}{\partial x} dx\right) dy\ dz - \rho u^2\ dy\ dz$$

$$= \frac{\partial(\rho u^2)}{\partial x} dx\ dy\ dz \quad [\text{kg m s}^{-2}] \quad (3)$$

The rate of flow of x component momentum through the face perpendicular to the y direction whose centre is R is u [m s^{-1}] times the mass flow through the face, $\rho v\ dx\ dz$ [kg s^{-1}]; that is,

$$\rho uv\ dx\ dz \quad [\text{kg m s}^{-2}]$$

The rate of flow of x component momentum through the opposite face whose centre is S is

$$\left(\rho uv + \frac{\partial(\rho uv)}{\partial y} dy\right) dx\ dz \quad [\text{kg m s}^{-2}]$$

and so the net rate of flow out of the CV through the faces with centres R and S is

$$\left(\rho uv + \frac{\partial(\rho uv)}{\partial y} dy\right) dx\ dz - \rho uv\ dx\ dz$$

$$= \frac{\partial(\rho uv)}{\partial y} dx\ dy\ dz \quad [\text{kg m s}^{-2}] \quad (4)$$

Similarly, the net rate of flow out of the CV through the faces normal to the z axis with centres T and U is

$$\frac{\partial(\rho uw)}{\partial z} dx\, dy\, dz \quad [\mathrm{kg\, m\, s^{-2}}] \quad (5)$$

Adding together (1.3), (1.4) and (1.5), the sum of the net rates of outflow of x component momentum is

$$\left[\frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z}\right] dx\, dy\, dz \quad [\mathrm{kg\, m\, s^{-2}}] \quad (6)$$

3 Body force terms

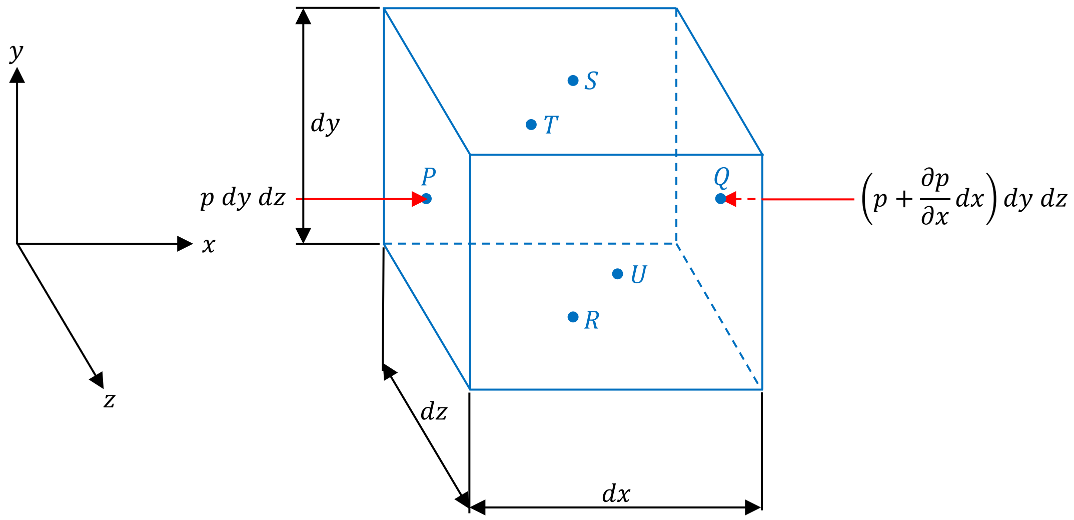
There are two types of forces acting on the fluid in the CV: body forces and surface forces. The simplest example of a body force is the gravitational force. The fluid in the CV is subject to a gravitational force equal to g [m s^{-2}], the acceleration due to gravity, times the mass of the fluid, $\rho\, dx\, dy\, dz$ [kg]; that is, $\rho g\, dx\, dy\, dz$ [kg m s^{-2}]. A body force is a vector, so in general it has three components f_x, f_y, f_z per unit mass [m s^{-2}]. The body force acting in the x coordinate direction is

$$\rho f_x\, dx\, dy\, dz \quad [\mathrm{kg\, m\, s^{-2}}] \quad (7)$$

4 Pressure term

Surface forces (i.e. forces on the imaginary surfaces of the CV) arise because of molecular stresses in the fluid. One kind of molecular stress is the pressure, which is present even in a fluid at rest, and it is normal to the surface on which it acts (see Figure 2). By definition, a positive pressure acts inwards.

Figure 2 Pressure acting in the x direction



The net force in the x direction acting on the CV is

$$-\left(p+\frac{\partial p}{\partial x}dx\right)dy\,dz+p\,dy\,dz$$

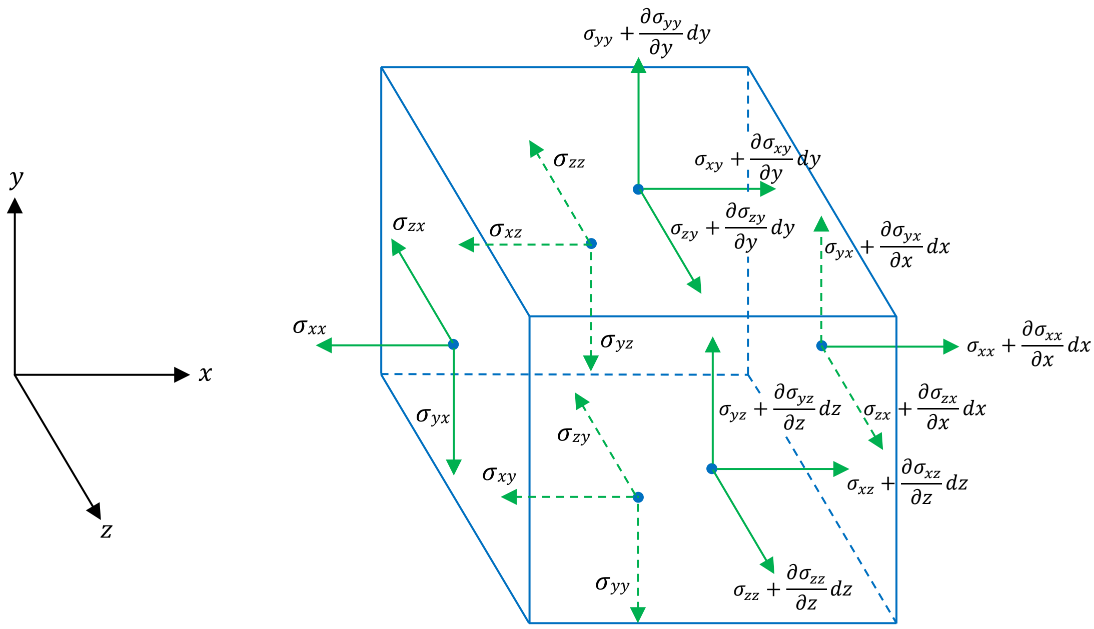
$$=-\frac{\partial p}{\partial x}dx\,dy\,dz \qquad [\mathrm{kg\ m\ s^{-2}}] \quad (8)$$

The faces with centres R to U do not contribute to the net force because the pressure on them is normal to the x direction.

5 Viscous stress terms

If a fluid element changes size or shape with time, viscosity creates further stresses that may act normal to a surface (a viscous normal stress) or tangentially (a viscous shear stress). We define the different components of viscous normal stress and viscous shear stress as shown in Figure 3. The first subscript of the symbol σ represents the direction of the stress and the second subscript represents the direction of the surface normal.

Figure 3 Viscous normal stresses and shear stresses in Cartesian coordinates



<u>Normal stresses</u>

By convention, an *outward* normal stress acting on the CV is positive.

<u>Shear stresses</u>

By convention, the shear stresses are taken as positive on the faces farthest from the origin. Thus a shear stress σ_{xy} acts in the positive x direction on the visible (upper) face perpendicular to the y axis and a corresponding shear stress acts in the negative x direction on the invisible (lower) face perpendicular to the y axis.

Referring to Figure 3, the net force in the x direction acting on the CV is

$$\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx\right) dy\, dz - \sigma_{xx} dy\, dz + \left(\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial y} dy\right) dx\, dz - \sigma_{xy} dx\, dz + \left(\sigma_{xz} + \frac{\partial \sigma_{xz}}{\partial z} dz\right) dx\, dy - \sigma_{xz} dx\, dy$$

$$= \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}\right) dx\, dy\, dz \quad [\mathrm{kg\ m\ s^{-2}}] \quad (9)$$

Collecting together the expressions (7), (8) and (9), the sum of the x components of the forces acting on the fluid in the CV is

$$\left(-\frac{\partial p}{\partial x}+\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+\frac{\partial \sigma_{xz}}{\partial z}+\rho f_x\right)dx\,dy\,dz \qquad [\mathrm{kg\,m\,s^{-2}}] \qquad (10)$$

6 Momentum equations in terms of stress

By substituting expressions (2), (6) and (10) into Eq. (1) and dividing by the volume of the CV, $dx\,dy\,dz$, we obtain the x component momentum equation for three-dimensional unsteady compressible flow in Cartesian coordinates in terms of stress:

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}+\frac{\partial(\rho uw)}{\partial z}=-\frac{\partial p}{\partial x}+\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+\frac{\partial \sigma_{xz}}{\partial z}+\rho f_x \quad [\mathrm{kg\ m^{-2}\ s^{-2}}] \quad (11)$$

We can derive equations for the y and z components of momentum in a similar way. The equations are, respectively:

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho vu)}{\partial x}+\frac{\partial(\rho v^2)}{\partial y}+\frac{\partial(\rho vw)}{\partial z}=-\frac{\partial p}{\partial y}+\frac{\partial \sigma_{yx}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{yz}}{\partial z}+\rho f_y \quad [\mathrm{kg\ m^{-2}\ s^{-2}}] \quad (12)$$

$$\frac{\partial(\rho w)}{\partial t}+\frac{\partial(\rho wu)}{\partial x}+\frac{\partial(\rho wv)}{\partial y}+\frac{\partial(\rho w^2)}{\partial z}=-\frac{\partial p}{\partial z}+\frac{\partial \sigma_{zx}}{\partial x}+\frac{\partial \sigma_{zy}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}+\rho f_z \quad [\mathrm{kg\ m^{-2}\ s^{-2}}] \quad (13)$$

7 Constitutive equations

For a variable-density Newtonian viscous fluid, the viscous normal stresses σ_{xx}, σ_{yy}, σ_{zz} [kg m^{-1} s^{-2}] and the viscous shear stresses σ_{xy}, σ_{yx}, σ_{xz}, σ_{zx}, σ_{yz}, σ_{zy} [kg m^{-1} s^{-2}] are given by

$$\sigma_{xx} = 2\mu \frac{\partial u}{\partial x} + \lambda \nabla \cdot \mathbf{V}$$

$$\sigma_{yy} = 2\mu \frac{\partial v}{\partial y} + \lambda \nabla \cdot \mathbf{V}$$

$$\sigma_{zz} = 2\mu \frac{\partial w}{\partial z} + \lambda \nabla \cdot \mathbf{V}$$

$$\sigma_{xy} = \sigma_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

$$\sigma_{yz} = \sigma_{zy} = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)$$

$$\sigma_{zx} = \sigma_{xz} = \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)$$

In these equations μ is the *shear viscosity* or *first viscosity* and λ is the *volume viscosity* or *bulk viscosity*. The *second viscosity* ζ is defined by

$$\zeta = \lambda + \frac{2}{3}\mu$$

The second viscosity ζ is often assumed to be zero, making the volume viscosity λ equal to $-2\mu/3$. This assumption seems to have a theoretical basis only in the case of an ideal monatomic gas. However, it is often carried over to both liquids and gases of all degrees of complexity.

The divergence term $\nabla \cdot \mathbf{V}$ is

$$\nabla \cdot \mathbf{V} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \right) \cdot (u\mathbf{i} + v\mathrm{j} + w\mathbf{k})$$

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \qquad [\mathrm{s}^{-1}]$$

8 Momentum equations in terms of velocity

By substituting the constitutive equations into Eqs. (11), (12) and (13) we obtain the three-dimensional unsteady compressible flow momentum equations in Cartesian coordinates:

x coordinate momentum

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}+\frac{\partial(\rho uw)}{\partial z}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}+\lambda\mathbf{\nabla}\cdot\mathbf{V}\right]$$

$$+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]+\rho f_x \quad [\mathrm{kg\,m^{-2}\,s^{-2}}] \quad (14)$$

y coordinate momentum

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho vu)}{\partial x}+\frac{\partial(\rho v^2)}{\partial y}+\frac{\partial(\rho vw)}{\partial z}=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]$$

$$+\frac{\partial}{\partial y}\left[2\mu\frac{\partial v}{\partial y}+\lambda\mathbf{\nabla}\cdot\mathbf{V}\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\rho f_y \quad [\mathrm{kg\,m^{-2}\,s^{-2}}] \quad (15)$$

z coordinate momentum

$$\frac{\partial(\rho w)}{\partial t}+\frac{\partial(\rho wu)}{\partial x}+\frac{\partial(\rho wv)}{\partial y}+\frac{\partial(\rho w^2)}{\partial z}=-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]$$

$$+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z}+\lambda\mathbf{\nabla}\cdot\mathbf{V}\right]+\rho f_z \quad [\mathrm{kg\,m^{-2}\,s^{-2}}] \quad (16)$$